Structured Handling of Scoped Effects

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@ESOP 2022

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This Talk

Scoped Effects for the Working Programmer

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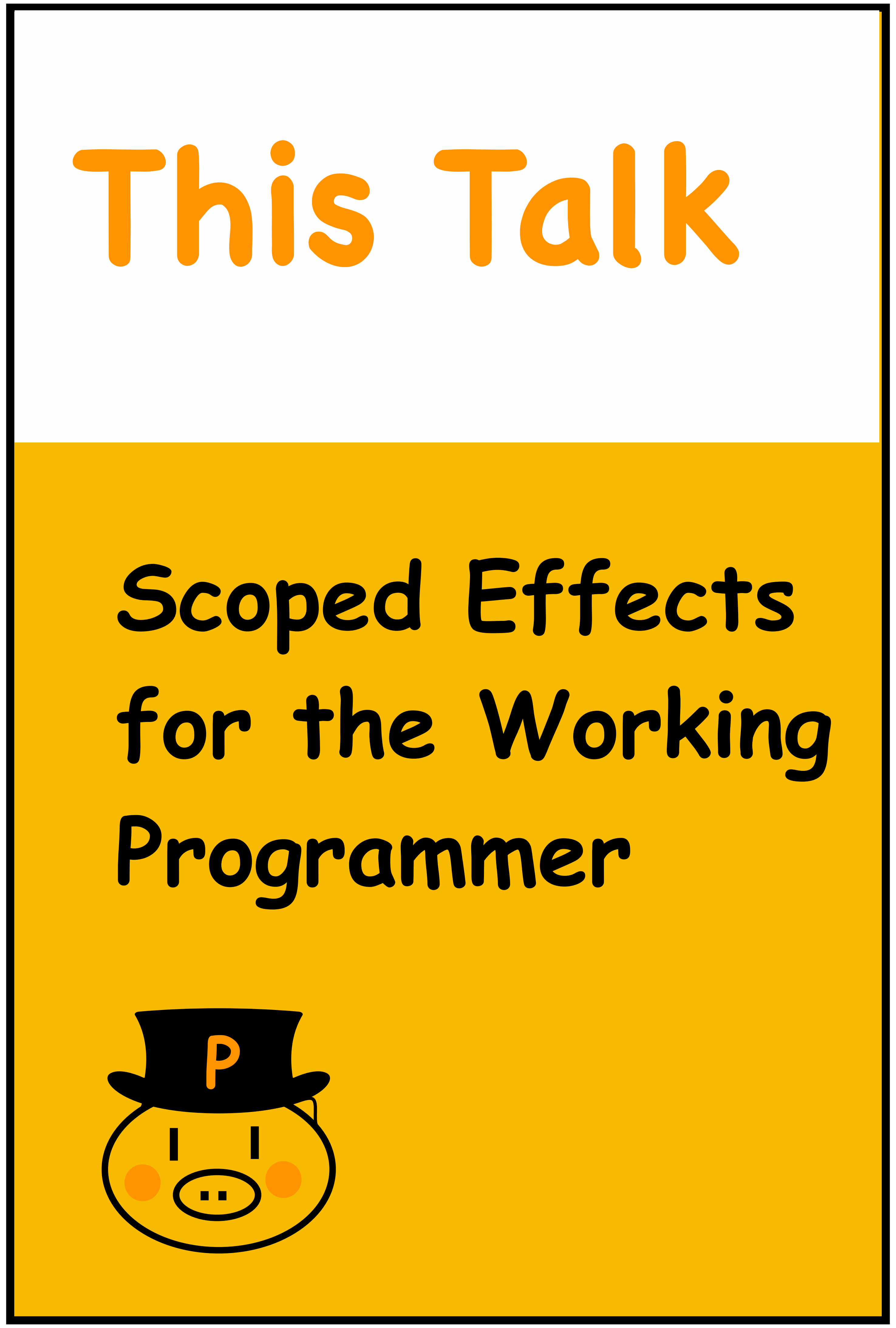

Scoped Effects
for the Working
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This Talk

Scoped Effects for the Working Programmer

More in the Paper

A Categorical Analysis of Our Approach

Algebraic Effects

A *computational effect* is modelled as an *algebraic theory.*

Example The effect of *mutable* **s**–*state* is modelled by

- two ***operations*** $\{\ \texttt{put : s} \rightsquigarrow \texttt{()},$
  $\texttt{get : ()} \rightsquigarrow \texttt{s}\ \}$

Algebraic Effects

A *computational effect* is modelled as an *algebraic theory.*

Example The effect of *mutable* **`s`**–*state* is modelled by

- two ***operations***
  ```
  { put : s  ⇝ (),
    get : () ⇝ s }
  ```
- several ***equations*** (pairs of *terms*) characterising **`put`** and **`get`**, such as
  ```
  do {put s; x ← get; k x} = do {put s; k s}
  ```
  …

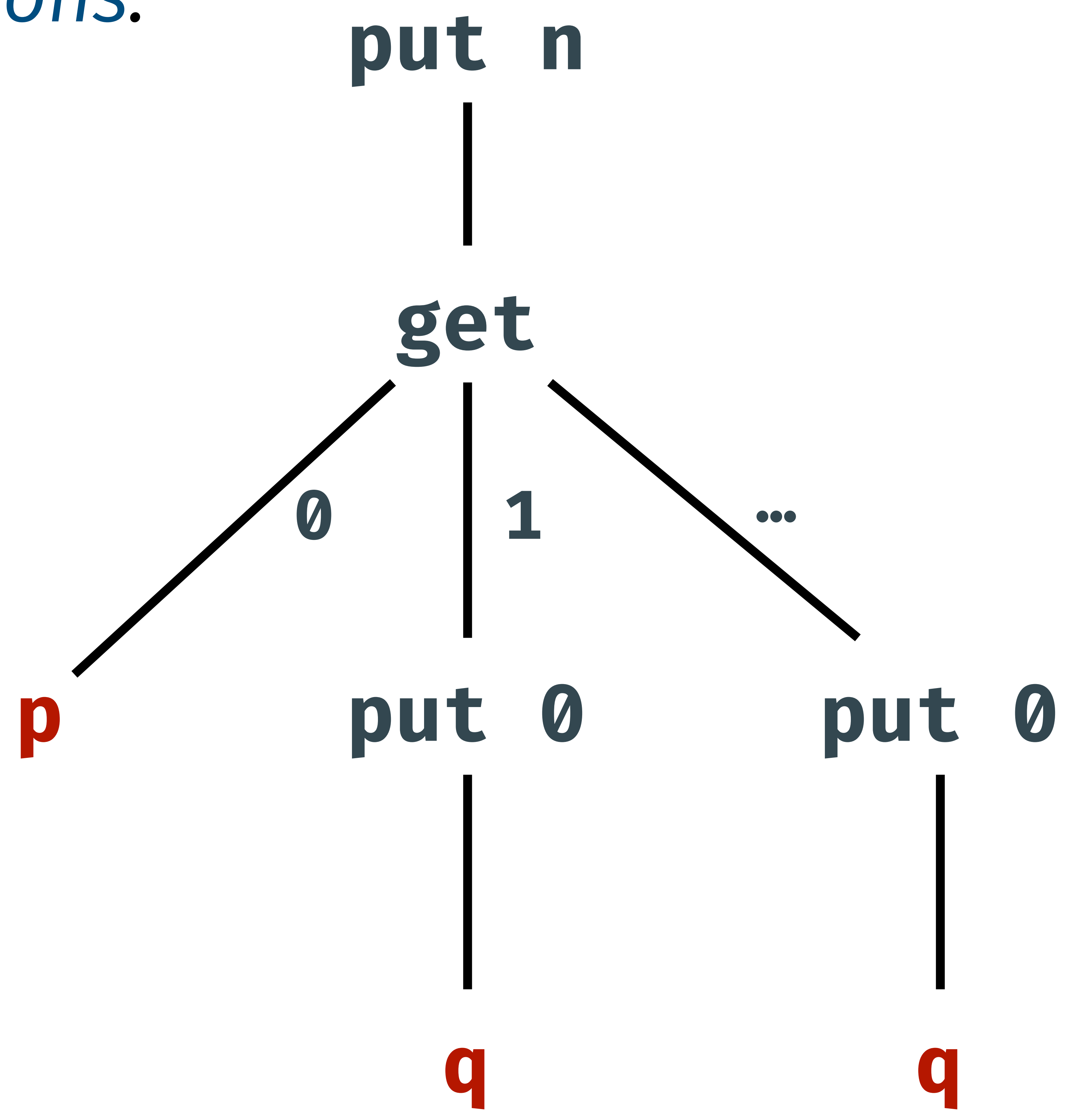
Terms of Operations

Terms of a theory are conceptually *trees of operations*.

Example *A term for a mutable* **`Int`***-state:*

```
do put n
   x ← get
   if x == 0
     then p
     else do put 0; q
```

$\simeq$



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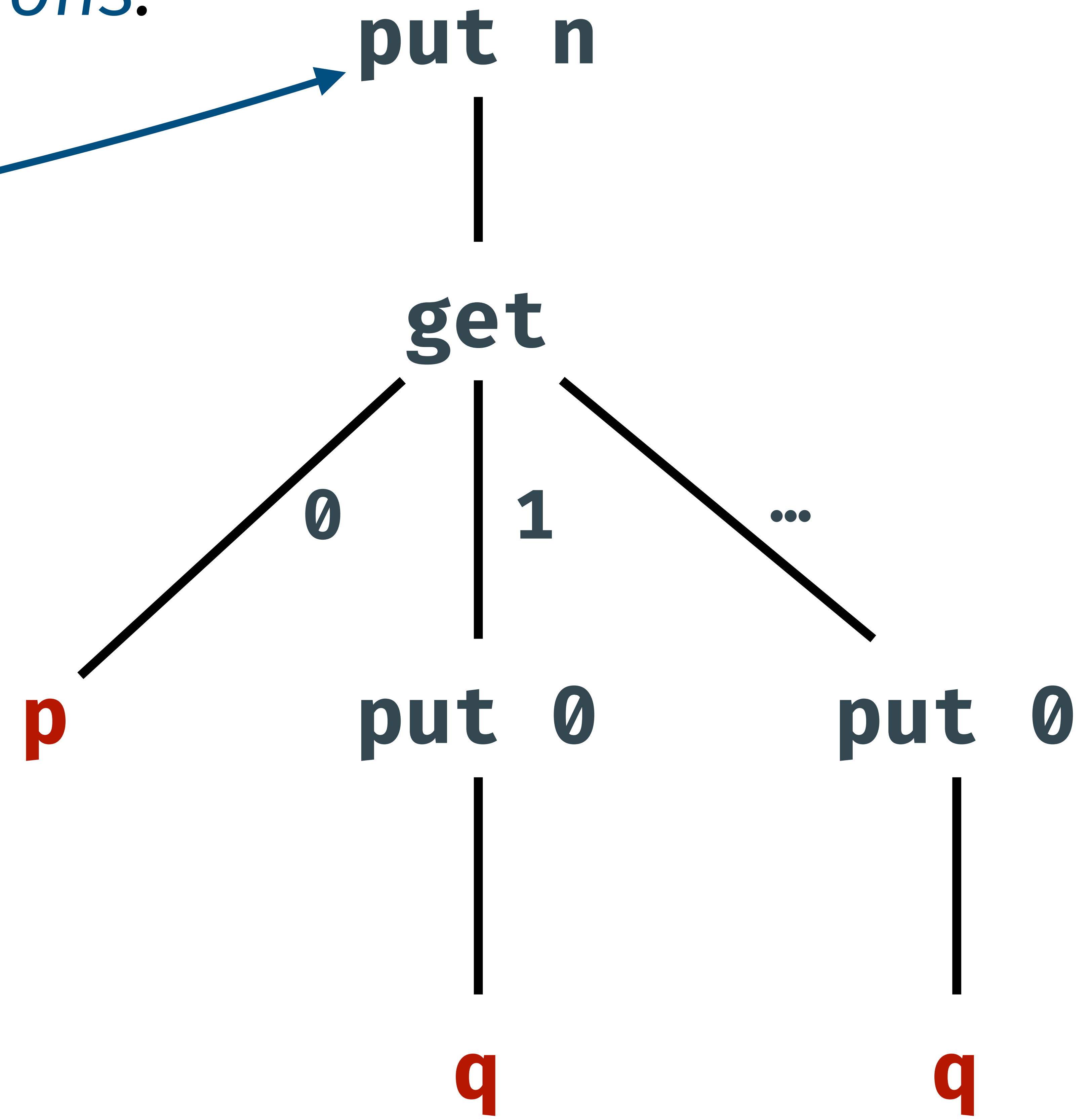

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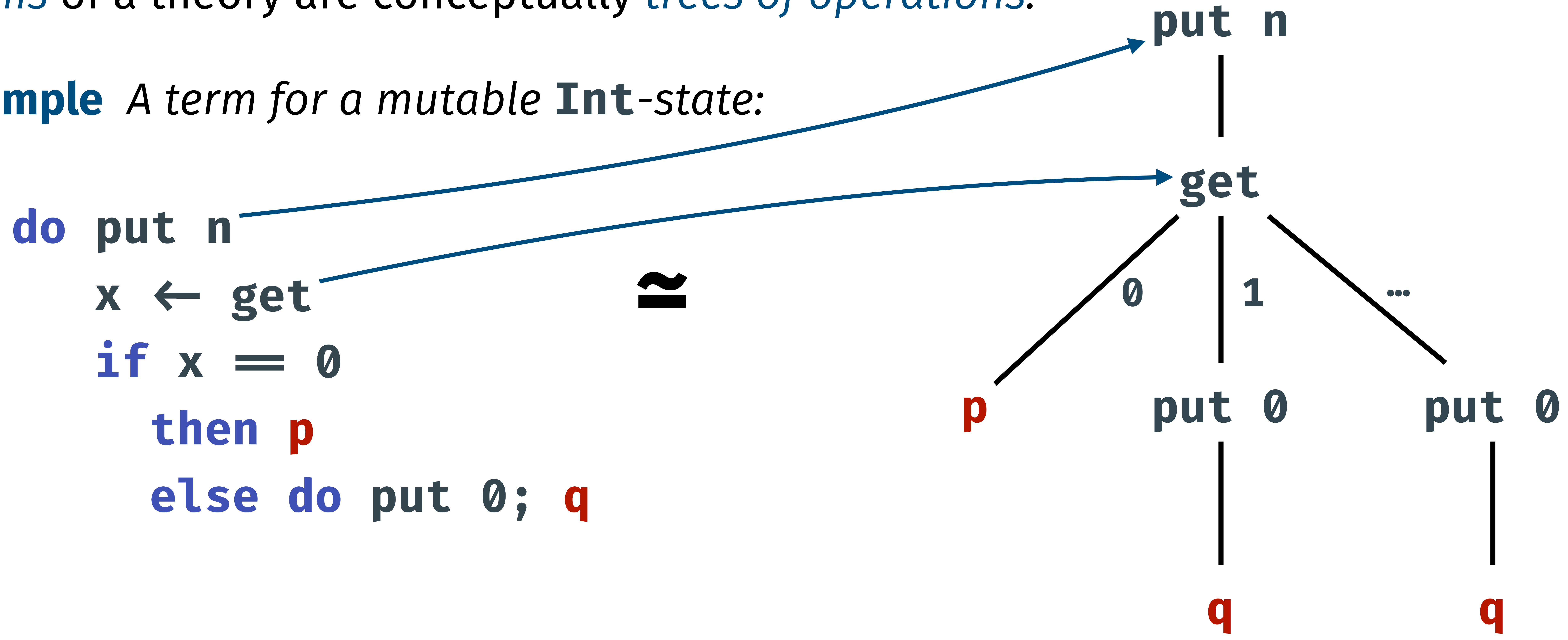

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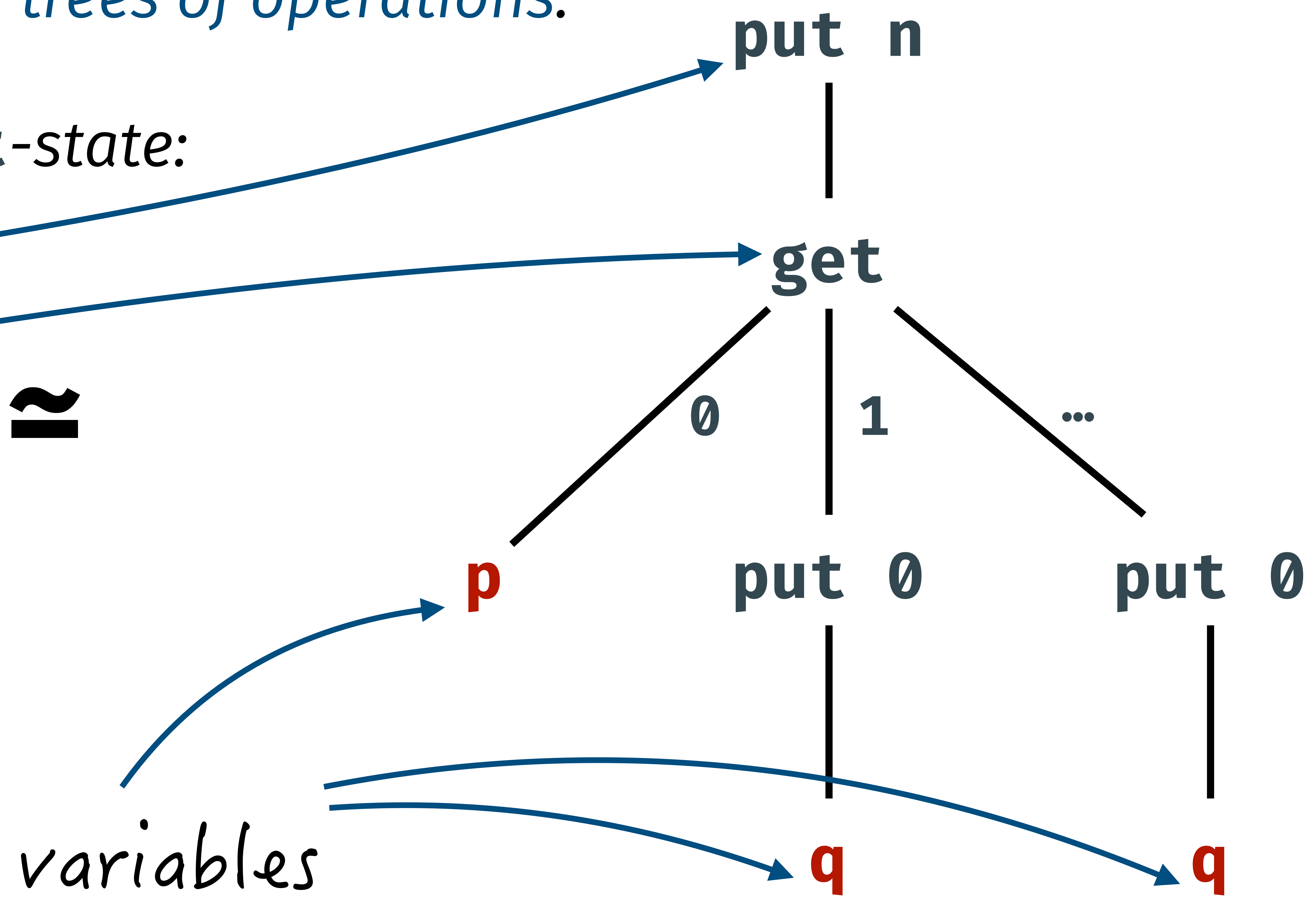

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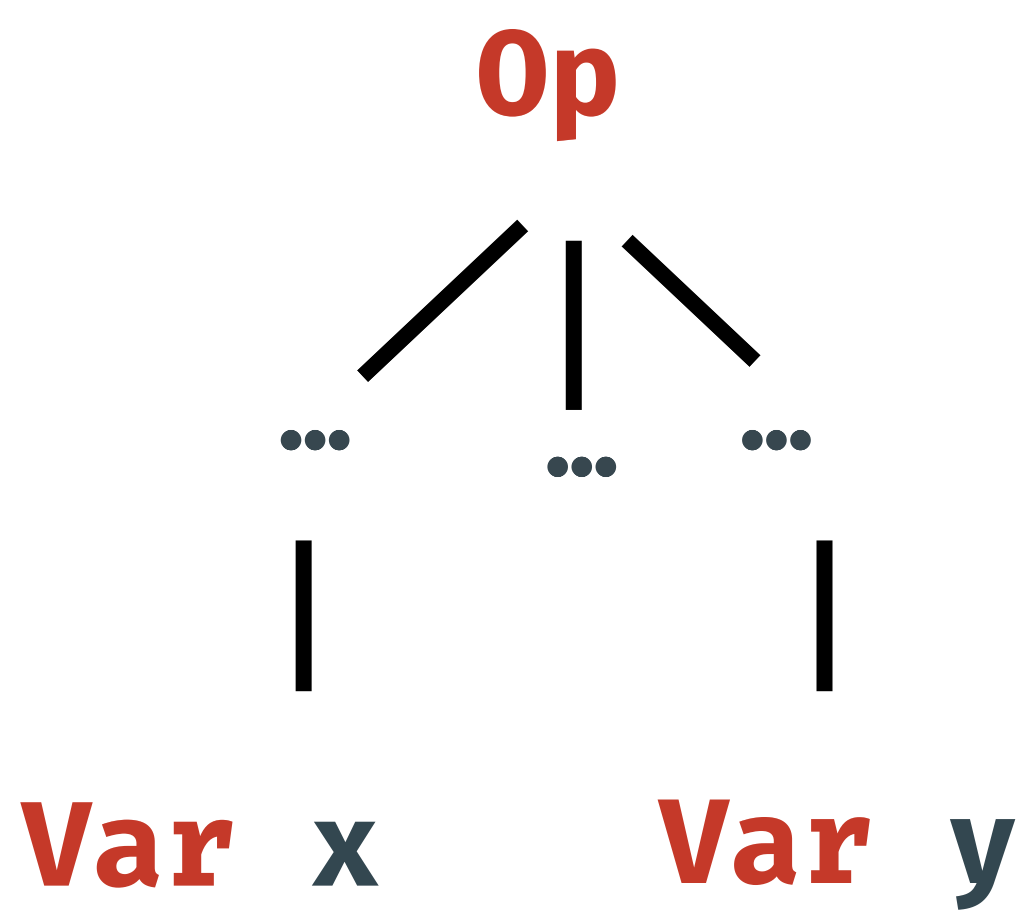

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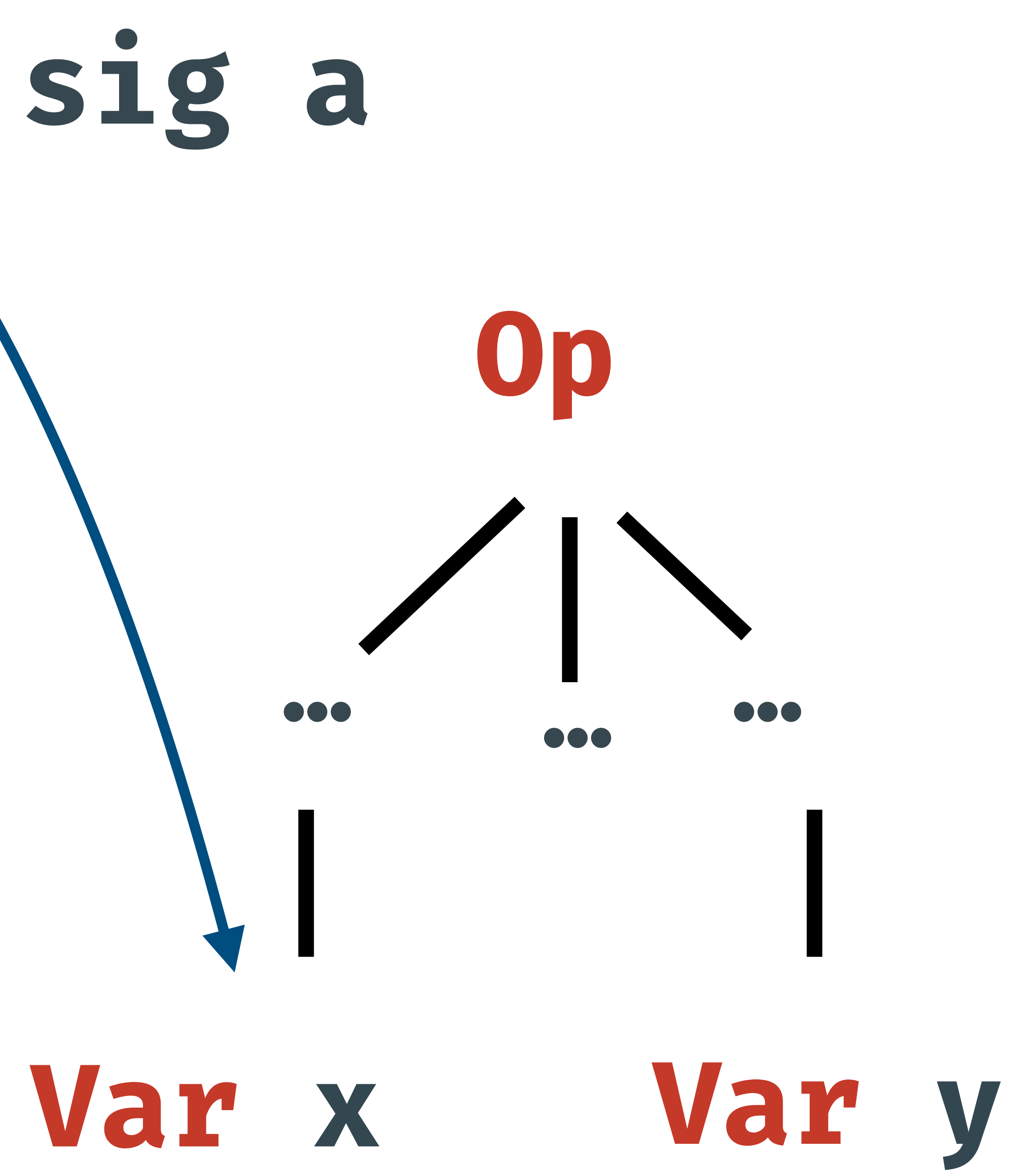

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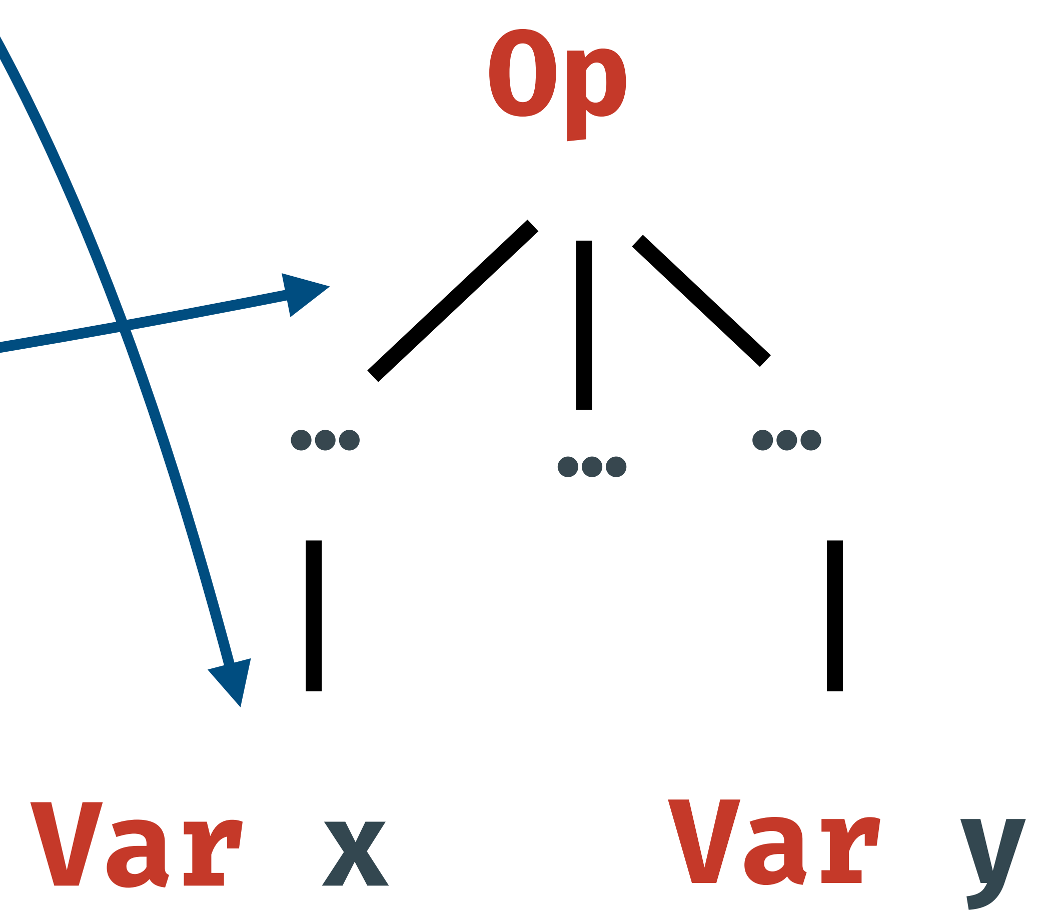

Operations are `sig`-branching internal nodes

Signature Functors

Signature of operations can be packaged into a datatype.

Example The signature for the effect of **`Int`**-*state* and *exception throw* is

```
data ES :: * → * where
  Put   :: Int → (()   → x) → ES x
  Get   :: ()  → (Int  → x) → ES x
  Throw :: ()  → (Void → x) → ES x
```

parameter type

result type

`Void` is the type with no constructors

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Example The signature for the effect of **`Int`**-*state* and *exception throw* is

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data ES :: * → * where
  Put   :: Int → x → ES x
  Get   :: (Int → x) → ES x
  Throw :: ES x
```

Term Model of Effectful Programs

Terms are a *syntactic model* of effectful computations.

Example A program involving **`Int`**-*state* and *exception* throwing:

```
safeDiv :: Int → Free ES Int
safeDiv n = Op (Get (λ s →
  if s ≡ 0
    then Op Throw
    else Op (Put (n / s)  (Var (n / s)))))
```

Variables are understood as return values

The Monad of Terms

We'd like to have *sequential composition* of (the term model of) computations, so we equip `Free sig` with a monad structure:

```
return :: a → Free sig a

(>>=) :: Free sig a → (a → Free sig b) → Free sig b
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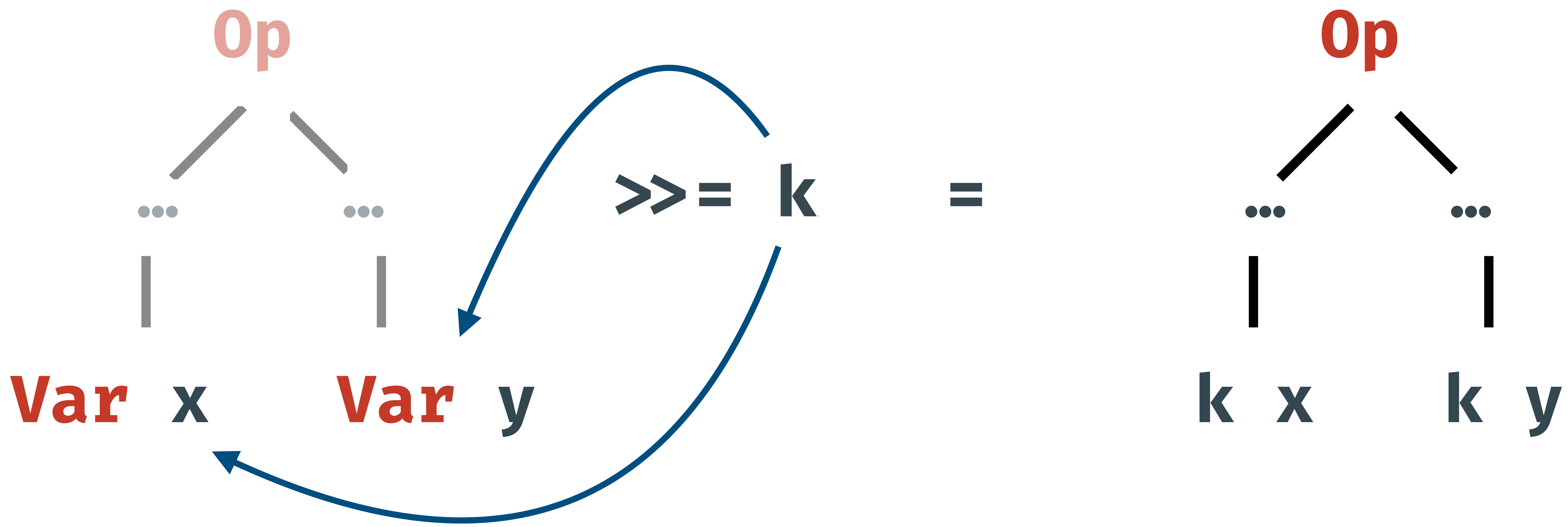

Effectful Programs with Free Monads

Example `safeDiv` is also sequential composition of smaller programs:

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Effectful Programs with Free Monads

Example `safeDiv` is also sequential composition of smaller programs:

```
safeDiv :: Int → Free ES Int
safeDiv n = get >>= λ s →
              if s ≡ 0
                then throw
                else put (n / s) >>= λ _ → return (n / s)

  where get     = Op (Get Var)
        put s   = Op (Put s (Var ()))
        throw   = Op Throw
```

Effectful Programs with Free Monads

Example `safeDiv` is also sequential composition of smaller programs:

```
safeDiv :: Int → Free ES Int
safeDiv n = do s ← get
               if s ≡ 0
                 then throw
                 else do put (n / s); return (n / s)

  where get     = Op (Get Var)
        put s = Op (Put s (Var ()))
        throw = Op Throw
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Effectful Programs with Free Monads

Example `safeDiv` is also sequential composition of smaller programs:

```
safeDiv :: Int → Free ES Int
safeDiv n = do s ← get
               if s ≡ 0
                 then throw
                 else do put (n / s); return (n / s)
```

`Free sig a` is just a *syntactic model* of effectful programs!

Handlers of Effects

Semantic models ("handlers") `<b :: *, f :: sig b→b>` *interpret ("handle")* programs with **`sig`**-operations:

```
handle :: (sig b → b) → (a → b) → (Free sig a → b)
```

How **`sig`**-operations act on the carrier **`b`**

How to turn a return value **`a`** into the carrier **`b`**

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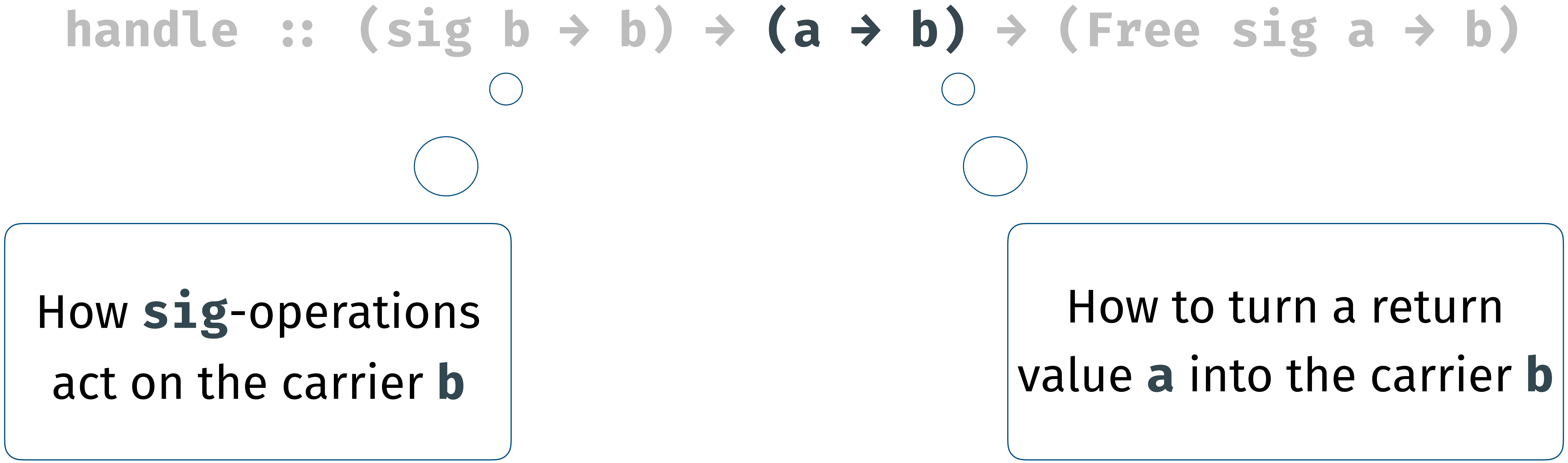

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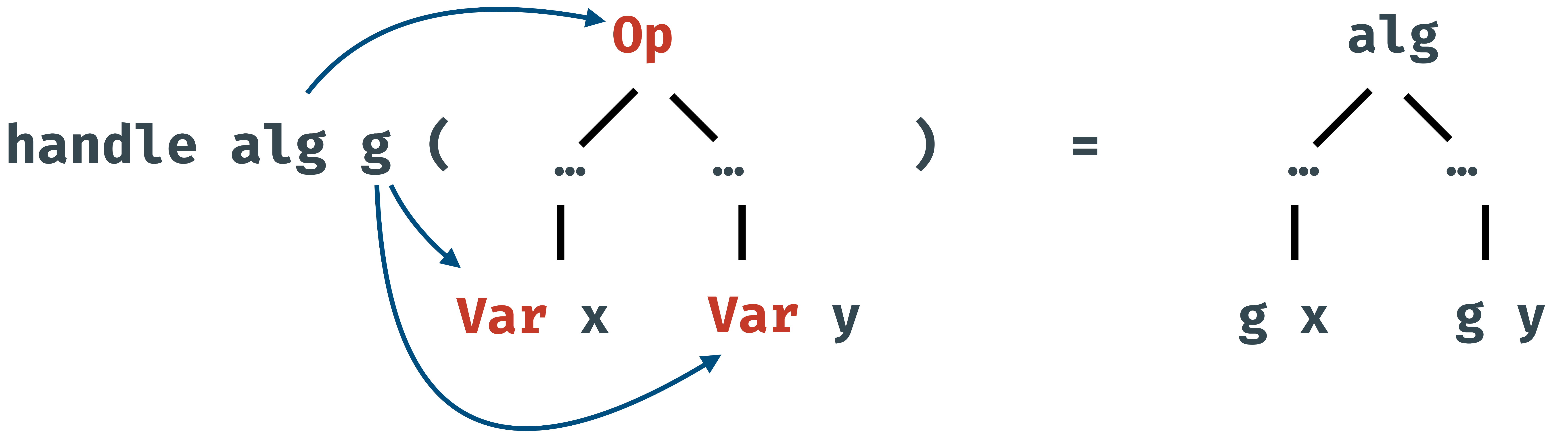

```
                         Op                                  alg
                        /  \                                /  \
handle alg g (        …      …        )    =              …      …
                      |      |                            |      |
                  Var x     Var y                        g x    g y
```

Handlers of Effects

Example Given a program `r :: Free ES a`, a handler `catchHdl r` that

- gives the ‘standard’ semantics to `Throw`, and
- leaves other operations unchanged:

```
catchHdl :: Free ES a
         → ES (Free ES a) → Free ES a
catchHdl r Throw = r
catchHdl r op    = Op op
```

Modularity of Handlers

Separating syntax from semantics allows different handlers of the same effect:

Example A non-standard handler of exception that *ignores* the recovery code `r`

```
catchHdl' :: Free ES a
          → ES (Free ES (Maybe a)) → Free ES (Maybe a)
catchHdl' r Throw = return Nothing
catchHdl' r op    = Call op
```

Non-Algebraic Operations

Why is exception throwing an operation but catching a handler?

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Why is exception throwing an operation but catching a handler?

If we model **`catch`** as an operation with **`Free`**, then

```
(catch p r) >>= k     ═     catch (p >>= k) (r >>= k)
```

by the definition of **`>>=`** for **`Free`**, but this equality is ***undesirable:***

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If we model `catch` as an operation with `Free`, then

```
(catch p r) >>= k      =      catch (p >>= k) (r >>= k)
```

by the definition of `>>=` for `Free`, but this equality is ***undesirable:***

The *scopes* for catching exceptions are different!

Non-Algebraic Operations

Although `catch` can be modelled as handlers, we ***lose*** the separation of syntax and semantics for `catch`:

Suppose we want a program that *morally* means

“
```
do x ← catch (safeDiv 5) (return 42)
   put (x + 1)
```
”

Non-Algebraic Operations

With different handlers, we write for `catchHdl`

```
do x ← handle (catchHdl (return 42)) return
                (safeDiv 5)
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```
do x ← handle (catchHdl (return 42)) return
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   put (x + 1)
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but for **`catchHdl’`** we write

```
do xMb ← handle (catchHdl' (return 42)) (return · Just)
                (safeDiv 5)
   case xMb of
     Nothing   → return Nothing
     (Just x ) → do r ← put (x + 1); return (Just r )
```

Non-Algebraic Operations

With different handlers, we write for **`catchHdl`**

```
do x ← handle (catchHdl (return 42)) return     :: Free ES a
              (safeDiv 5)
   put (x + 1)
```

but for **`catchHdl'`** we write

```
do xMb ← handle (catchHdl' (return 42)) (return · Just) :: Free ES (Maybe a)
                (safeDiv 5)
   case xMb of
     Nothing   → return Nothing
     (Just x ) → do r ← put (x + 1); return (Just r))
```

Scoped Effects

We want to write *syntactic non-algebraic operations* and interpret them differently.

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do x ← catch (safeDiv 5) (return 42)
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Cause Handlers model the syntax and semantics of `catch` at the same time!

Scoped Effects

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Solution Separate syntax and semantics.

Scoped Effects

We want to write *syntactic non-algebraic operations* and interpret them differently.

“
```
do x ← catch (safeDiv 5) (return 42)
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```
”

Cause Handlers model the syntax and semantics of `catch` at the same time!

Solution

- Generalising `Free` to non-algebraic (“*scoped*”) operations [Wu et al. 2014];
- Finding nice ways to handle them (*contribution of this paper*).

Syntax of Scoped Effects

Extending **`Free`** to accommodate scoped operations:

```
data Free f a :: * where
  Var :: a → Free f a
  Op  :: f (Free f a) → Free f a
```

Syntax of Scoped Effects

Extending `Free` to accommodate scoped operations:

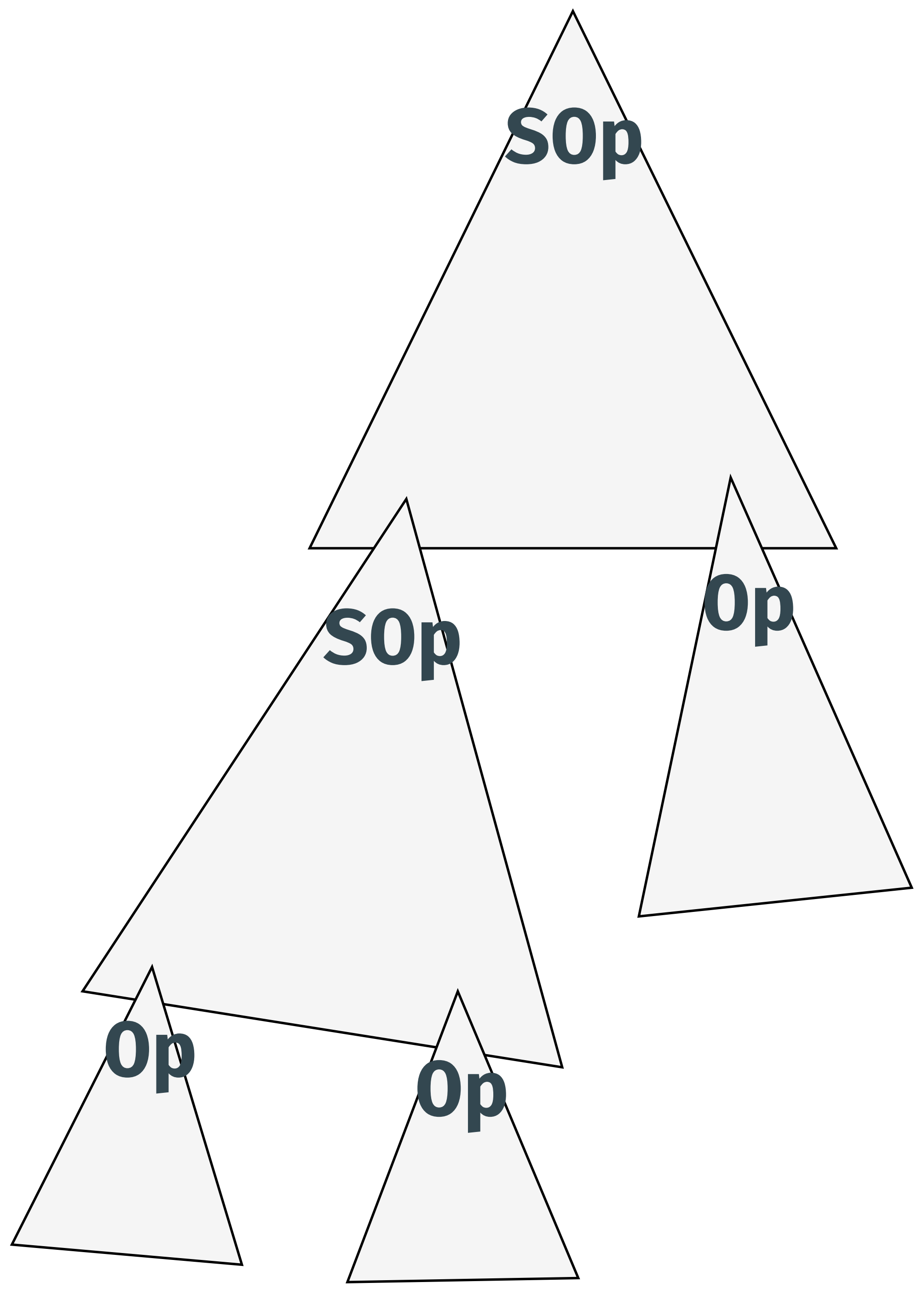
```
data FreeS f g a :: * where
  Var :: a → FreeS f g a
  Op  :: f (FreeS f g a) → FreeS f g a
  SOp :: g (FreeS f g (FreeS f g a)) → FreeS f g a
```

`f`: *signature of algebraic operations*
`g`: *signature of scoped operations*

Syntax of Scoped Effects

Intuition `Free f` are trees, while `FreeS f g` are *nested trees*:

- Boundary of a tree is the scope of an scoped operation



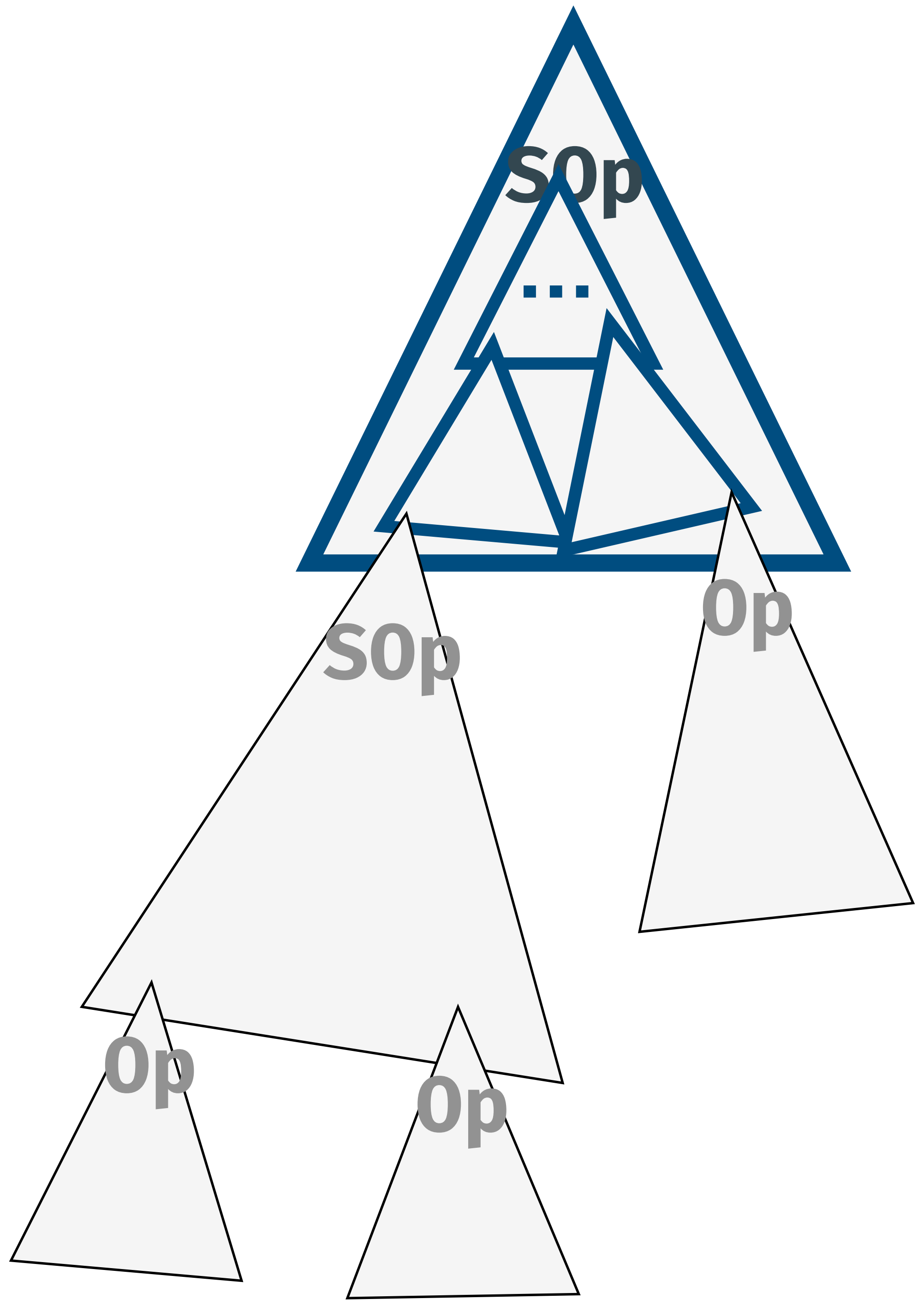
Syntax of Scoped Effects

Intuition **`Free f`** are trees, while **`FreeS f g`** are ***recursively*** *nested trees:*

- Boundary of a tree is the scope of an scoped operation

- Trees themselves can be nested trees, i.e. scoped operations can be nested.

```
catch (catch p h >>= k) h'
  >>= k'
```



Handlers of Scoped Effects

What are the handlers of scoped operations?

Proposal 1 Treating them as algebraic effects with recursion

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Handlers of Scoped Effects

What are the handlers of scoped operations?

Proposal 1 Treating them as algebraic effects with recursion

```
data FreeS f g a :: * where
  Var :: a → FreeS f g a
  Op  :: (f + g ∘ FreeS f g) (FreeS f g a) → FreeS f g a
```

Scoped operations are treated as algebraic operations whose signature is *recursively* defined

Handlers of Scoped Effects

What are the handlers of scoped operations?

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```
opB   :: f c → c          sopB  :: g (FreeS f g c) → c
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Handlers of Scoped Effects

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```
opB   ::  f  c → c          sopB  ::  g  (FreeS  f  g  c) → c
```

Problem `sopB` has *too much* freedom on how to use `FreeS f g`

Proposal of This Paper

A *functorial algebra* for algebraic signature **f** and scoped signature **g** has

- and a type **c :: *** equipped with

```
opB  ::  f c → c
sopB ::  g (h c) → c
```

- A functor **h :: *→*** equipped with

```
varE :: ∀x. x → h x
opE  :: ∀x. f (h x) → h x
sopE :: ∀x. g (h (h x)) → h x
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```

which gives rise to a handling function:

```
handle  ::  FunctorialAlg h c
        →  (a → c) → FreeS f g a → c
```

Some Examples

- *Exception throwing and catching* handled by `<Maybe, Maybe a, …>`
- *Explicit nondeterminism with scoped search strategies* like

```
    bfs (or (dfs (or  ... ))
             (or x y))
```

  handled by `<x ⟼ ([x], [[x]]),  [a], …>`

- *Parallel composition handled by a resumption monad.*

What Else in the Paper

THM There is an adjunction between functorial algebras and the category $\mathbb{C}$ (for pure values)

$$\begin{tikzcd}
\mathit{Fn}\text{-}\mathit{Alg} \arrow[r, shift right, "U_{\mathit{Fn}}"'] \arrow[r, phantom, "\perp"] & \mathit{Endo}_f(\mathbb{C}) \times \mathbb{C} \arrow[l, shift right, "\mathit{Free}_{\mathit{Fn}}"'] \arrow[r, shift right, "\downarrow"'] \arrow[r, phantom, "\perp"] & \mathbb{C} \arrow[l, shift right, "\uparrow"'] \arrow[loop right, "T"]
\end{tikzcd}$$

whose induced monad T is isomorphic to **`FreeS f g`**.

What Else in the Paper

Functorial algebras are compared with two other adjunctions for handling scoped effects: *indexed algebras* and *Eilenberg-Moore algebras*.

THM There are comparison functors between these adjunctions. Thus all these models have equal expressivity.

$$\begin{tikzcd}
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\mathbb{C}^P \arrow[rr, shift right=1.5ex, "U^P"'] \arrow[rr, phantom, "\perp"] \arrow[uu, bend left=60, "K^{\mathtt{EM}}_{\mathtt{Fn}}"] & & \mathbb{C} \arrow[ll, shift right=1.5ex, "\mathit{Free}^P"']
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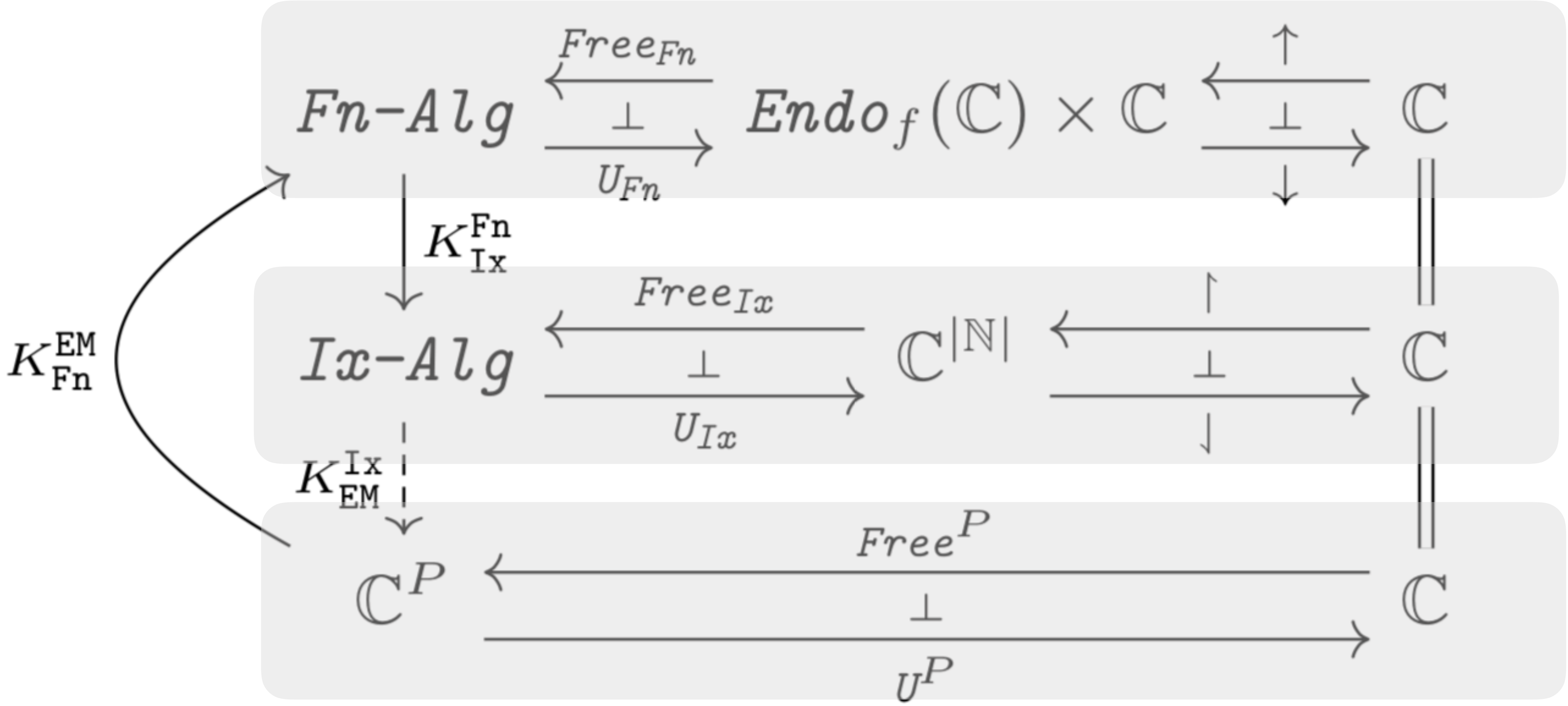

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\end{tikzcd}$$

Take-Home Messages

📬🏠

- Non-algebraic operations need not to be handlers.
- They can be operations and handled in a structural way.

Back up slides

Scoped Scoped Operations?

We indeed can make a further generalisation:

```
SSOp :: g (FreeS f g (FreeS f g (FreeS f g a)))
     → FreeS f g a
```

corresponding to operations that look like

```
op ({ P  }{x. Q },
    { P' }{y. Q’}, … )
```

Example: explicit substitution `subst(P)(x. Q)`, but not too many.

Connections to Delimited Control

Can we implement scoped operations with shift/reset?

- Sounds plausible.

Are shift/reset scoped operations?

- Interesting direction. We need to develop scoped operations on *parameterised monads*, since `shift` and `reset` are not operations on ordinary monad `Cont r` but on parametric monad `Cont`.

```
shift :: ((a → r) → Cont r r) → Cont r a
reset :: Cont r r → Cont w r
```