Functional Data Structures in Monoidal Categories

(work in progress)

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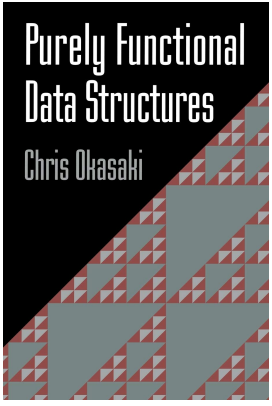
Context

One of the greatest achievements of functional programming is the invention of many purely functional data structures.

Purity implies *persistence*, which is important in text editors, version control systems, databases, …



[Oka98]

Sequences

Sequences of data are ubiquitous in programming. A data structure for sequences is

$$
\begin{array}{ll}
S : \mathit{Type} \to \mathit{Type} & \mathit{nil} : S\ a \\
\mathit{cons} : a \to S\ a \to S\ a & \mathit{matchL} : b \to (a \to S\ a \to b) \to b \\
\mathit{snoc} : S\ a \to a \to S\ a & \mathit{matchR} : b \to (S\ a \to a \to b) \to b \\
(+\!\!+) : S\ a \to S\ a \to S\ a & \mathit{splitAt} : \mathit{Int} \to S\ a \to (S\ a, S\ a)
\end{array}
$$

satisfying certain equations.

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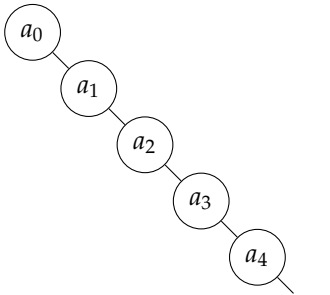
\end{array}
$$

satisfying certain equations.

More operations (*head*, *tail*, *lookup*, *fold*, …) can be defined from the primitives above.

Cons lists

data *List* (*a* : *Type*) **where**
  Nil : *List a*
  Cons : *a* → *List a* → *List a*

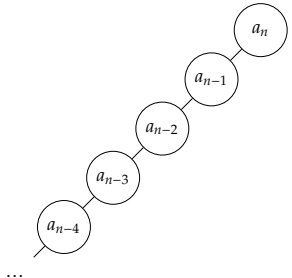


- Constant time for *cons* and *matchL*.
- Linear time for other operations.

Snoc lists

data *SnocList* (*a* : *Type*) **where**
  Nil : *SnocList a*
  Snoc : *SnocList a* → *a* → *SnocList a*

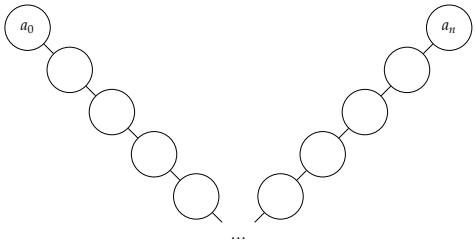
- Constant time for *snoc* and *matchR*.
- Linear time for other operations.



Batched Queue

data *BatchedQueue* (*a* : *Type*) **where**
Q : *List a* → *SnocList a* → *BatchedQueue a*

- Constant time for *cons* and *snoc*.
- Amortised constant time for *matchL* (in the partially persistent setting).

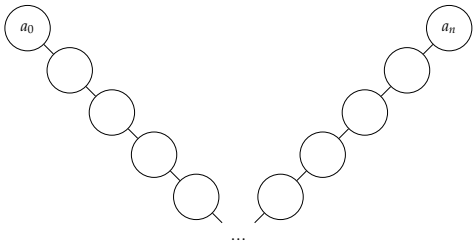


Banker Queue

data *BankerQueue* (*a* : *Type*) **where**
Q : *Int* → *List a* → *Int* → *SnocList a*
→ *BankerQueue a*

assuming laziness [Oka98].

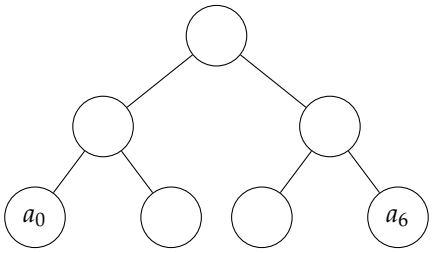
- Constant time for *cons* and *snoc*.
- Amortised constant time for *matchL* (in the fully persistent setting).



Length of the front half ⩾ length of the rear half.

Balanced Binary Trees

```
data AVLTrees (a : Type) where
  Nil : AVLTrees a
  Node : Int    -- tree height
       → AVLTrees a
       → a
       → AVLTrees a
       → AVLTrees
```



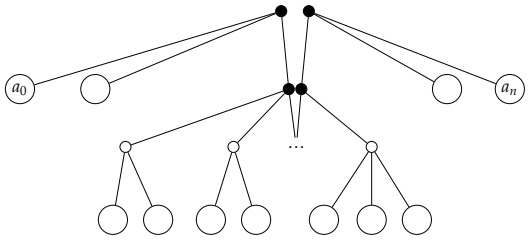
- Logarithmic time for all operations.

Finger Trees

data *FingerTrees* (*a* : *Type*) **where**

…

- Amortised constant time for *cons*, *snoc*, *matchL*, *matchR*.
- Logarithmic time for ⧺ and *splitAt*.

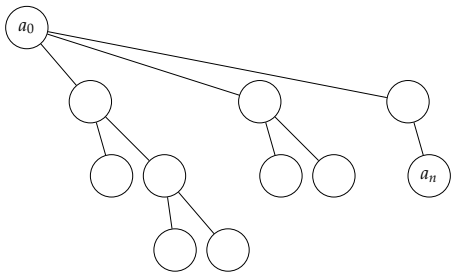


Catenable Lists

data *CList* (*a* : *Type*) **where**
Nil : *CList a*
Cons : *a* → *Queue* (*CList a*) → *CList a*

assuming laziness and some *Queue* supporting *snoc* and *matchL* in $O(1)$.

- Constant time for *cons*, *snoc*, ++, *matchL* [Oka98].
- Linear time for *matchR* and *splitAt*.



Essentially a rose tree with children of a node stored with a queue.

Main Idea

Observation

These data structures usually are implemented in languages with cartesian contexts (ML, Haskell, Agda, Idris, …), but *they use data linearly and orderly*.

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For example, concatenating two lists is

$$
\begin{aligned}
&(+\!\!+) : \mathit{List}\ a \longrightarrow \mathit{List}\ a \longrightarrow \mathit{List}\ a \\
&\mathit{Nil} \qquad\quad +\!\!+\ \mathit{ys} = \mathit{ys} \\
&\mathit{Cons}\ x\ \mathit{xs} +\!\!+\ \mathit{ys} = \mathit{Cons}\ x\ (\mathit{xs} +\!\!+\ \mathit{ys})
\end{aligned}
$$

After all, why would a sequence structure discard/reorder/duplicate its data?

Main Idea

Idea

Linear type theories have semantics in suitable monoidal categories. Let's *interpret these data structures in suitable monoidal categories* to obtain new data structures!

Motivating Example

I am mostly interested in the monoidal category of *endofunctors and composition*.

$$
\left[\!\!\left[
\begin{array}{l}
\text{types } A \\
\text{terms } t \\
\text{linear product } A \otimes B \\
\text{lists } \mathit{List}\ A \cong 1 + A \otimes \mathit{List}\ A \\
\text{concatenation } (+\!\!+)
\end{array}
\right]\!\!\right]
\quad = \quad
\begin{array}{l}
\text{endofunctors } F \\
\text{natural transformation } \phi : F \dot{\rightarrow} G \\
\text{functor composition } F \circ G \\
\text{free monads } \mathit{Free}\ F \cong \mathit{Id} + F \circ \mathit{Free}\ F \\
\text{multiplication } \mu
\end{array}
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$$

Question

Can we obtain efficient implementations of free monads from efficient sequence data structures?

A Strange New World

Monoidal categories are a strange new world for doing data structures.

Strangeness 1

Lists and free monoids may differ when the monoidal category isn't closed.

To interpret the definition of $+\!\!+$, we need not only the universal property of the initial algebra $\mu X.\ I + a \otimes X$ but also the right adjoint $\mathit{List}\ a \multimap -$ to $- \otimes \mathit{List}\ a$.

$$
\begin{aligned}
&(+\!\!+) : \mathit{List}\ a \to (\mathit{List}\ a \to \mathit{List}\ a) \\
&\mathit{Nil} \qquad\quad +\!\!+\ ys = ys \\
&\mathit{Cons}\ x\ xs +\!\!+\ ys = \mathit{Cons}\ x\ (xs +\!\!+\ ys)
\end{aligned}
$$

A Strange New World

Example

Let $\langle \textsc{Mon}, \times, 1 \rangle$ be the cartesian monoidal category of monoids in $\textsc{Set}$. It is *not closed*.

- The free monoid over $A = \langle |A|, \cdot_A, \epsilon_A \rangle \in \textsc{Mon}$ is the free commutative monoid on A (following an Eckmann-Hilton argument).

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Example

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- The free monoid over $A = \langle |A|, \cdot_A, \epsilon_A \rangle \in \textsc{Mon}$ is the free commutative monoid on A (following an Eckmann-Hilton argument).
- The list object $\mu X.\ 1 + A \times X$ over A has underlying set *List* $|A|$, unit [], and element-wise multiplication using $\cdot_A$ and ϵ_A.

Strangeness 2

The costs of accessing the two components of the monoidal product $A \otimes B$ may differ.

In the monoidal category of *Type*-endofunctors and composition,
$x : A, y : B \vdash (f\ x, y) : A' \otimes B$ is interpreted as $f_{BX} : A(BX) \to A'B(X)$ while
$x : A, y : B \vdash (x, g\ y) : A \otimes B'$ is interpreted as $A\ g_X : A(BX) \to A(B'X)$.

If A is a container functor, $A\ g_X$ needs to traverse over the container to apply g_X.

Strangeness 3

Isomorphic monoidal structures may have different cost characteristics.

Under suitable foundational assumptions, functor composition $A \circ B$ has the isomorphic co-Yoneda form $\int^{X:Type} AX \times (X \to B-)$, for which we do not need to map over A to access B.

A Strange New World

Strangeness 4

Snoc-lists and *Cons*-lists may differ, depending on how monoidal products $\otimes$ distributes over coproducts.

In the monoidal category of *Type*-endofunctors and composition, we have $(A + B) \circ C \cong (A \circ C) + (B \circ C)$ but not $A \circ (B + C) \circ (A \circ B) + (A \circ C)$.

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Consequently, letting $\mathit{Iter}\ A = \mu X.\ \mathit{Id} + (X \circ A)$,

$$\begin{aligned}
\mathit{Iter}\ A &\cong \mathit{Id} + \mathit{Iter}\ A \circ A \cong I + (I + \mathit{Iter}\ A \circ A) \circ A \\
&\cong \mathit{Id} + A + \mathit{Iter}\ A \circ A \circ A \\
&\cong \mathit{Id} + A + (A \circ A) + (A \circ A \circ A) + \cdots \\
&\ncong \mathit{Free}\ A
\end{aligned}$$

Strangeness 5

Case analysis of coproducts may only work under certain contexts, depending on how monoidal products $\otimes$ distributes over coproducts.

In an ordered linear context $(\Gamma_l, x : A + B, \Gamma_r)$, the semantics of pattern matching x needs distributivity of $\otimes$ over $+$, which we may not have.

Data Structures in Monoidal Categories

Cons lists and snoc lists work[1] in any monoidal categories with coproducts, relevant initial algebras, right adjoints $A \multimap -$ to $- \otimes A$ for all A.

[1] except *matchR* for cons lists, *cons*/$\dashv$ for snoc lists

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It's a mis-conception that a snoc list is a reversed list. $\mathit{reverse} : \mathit{List}\ a \longrightarrow \mathit{List}\ a$ needs symmetry but *yank* does not.

$$
\begin{aligned}
&\mathit{yank} : \mathit{SnocList}\ a \longrightarrow \mathit{List}\ a \\
&\mathit{yank}\ sx = \mathit{go}\ xs\ \mathit{Nil}\ \textbf{where} \\
&\quad \mathit{go} : \mathit{SnocList}\ a \longrightarrow \mathit{List}\ a \longrightarrow \mathit{List}\ a \\
&\quad \mathit{go}\ \mathit{Nil}\ ys \qquad\quad = ys \\
&\quad \mathit{go}\ (\mathit{Snoc}\ sx\ x)\ ys = \mathit{go}\ sx\ (\mathit{Cons}\ x\ ys)
\end{aligned}
$$

[1] except *matchR* for cons lists, *cons*/$+\!\!+$ for snoc lists

Data Structures in Monoidal Categories

Banker queues work[2] in any monoidal category with *copowers*, coproducts, relevant initial algebras, right adjoints $A \multimap -$ to $- \otimes A$ for all A.

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Data Structures in Monoidal Categories

Banker queues work[2] in any monoidal category with *copowers*, coproducts, relevant initial algebras, right adjoints $A \multimap -$ to $- \otimes A$ for all A.

The size integers are used non-linearly. We should interpret them as copowers.

$$
\begin{aligned}
&\textbf{data}\ \mathit{BankerQueue}\ (a : \mathit{Type})\ \textbf{where}\\
&\quad Q : \mathit{Int} \to \mathit{List}\ a \to \mathit{Int} \to \mathit{SnocList}\ a \to \mathit{BankerQueue}\ a\\
&q : \mathit{Int} \to \mathit{List}\ a \to \mathit{Int} \to \mathit{SnocList}\ a \to \mathit{BankerQueue}\ a\\
&q\ \mathit{lenF}\ f\ \mathit{lenR}\ r = \textbf{if}\ \mathit{lenF} < \mathit{lenR}\\
&\quad \textbf{then}\ Q\ (\mathit{lenF} + \mathit{lenR})\ (f +\!\!+ \mathit{yank}\ r)\ 0\ \mathit{Nil}\\
&\quad \textbf{else}\ \ Q\ \mathit{lenF}\ f\ \mathit{lenR}\ r
\end{aligned}
$$

[2] *snoc*, *cons*, *matchL* works but not the others

Data Structures in Monoidal Categories

Balanced binary trees and finger trees work in any monoidal category with copowers, coproducts, relevant initial algebras, and *bi-closedness* (adjunctions $- \otimes A \dashv A \multimap -$ and $A \otimes - \dashv A \rightarrowtail -$ for all A). Symmetry is not necessary!

Data Structures in Monoidal Categories

Balanced binary trees and finger trees work in any monoidal category with copowers, coproducts, relevant initial algebras, and *bi-closedness* (adjunctions $- \otimes A \dashv A \multimap -$ and $A \otimes - \dashv A \rightarrowtail -$ for all A). Symmetry is not necessary!

Bi-closedness implies distributivity of $\otimes$ over $+$, allowing us to pattern match variables in any context:

$$
\begin{aligned}
&\mathit{rotateL}, \mathit{rotateR} : \mathit{AVLTree}\ a \longrightarrow \mathit{AVLTree}\ a \\
&\mathit{rotateL}\ (\mathit{Node}\ \_\ p\ x\ (\mathit{Node}\ \_\ q\ y\ r)) = \mathit{node}\ (\mathit{node}\ p\ x\ q)\ y\ r \\
&\mathit{rotateR}\ (\mathit{Node}\ \_\ (\mathit{Node}\ \_\ p\ x\ q)\ y\ r) = \mathit{node}\ p\ x\ (\mathit{node}\ q\ y\ r)
\end{aligned}
$$

With a small tweak, catenable lists work[3] in any monoidal category with copowers, coproducts, relevant initial algebras, $- \otimes A \dashv A \multimap -$. *No need for bi-closedness!*

$$
\begin{aligned}
&(+\!\!+) : \mathit{CList}\ a \longrightarrow \mathit{CList}\ a \longrightarrow \mathit{CList}\ a \\
&\quad \text{-- } \mathit{xs} +\!\!+ \mathit{Nil} = \mathit{xs} \\
&\mathit{Nil} +\!\!+ \mathit{ys} = \mathit{ys} \\
&C\ x\ \mathit{xss} +\!\!+ \mathit{ys} = C\ x\ (\mathit{snoc}\ \mathit{xss}\ \mathit{ys})
\end{aligned}
$$

We can't pattern match the second argument when xs is in the context. (This has complexity impact though).

[3] except for *matchR*/*splitAt*

Some Monoidal Categories

- *Symmetric closed*: cartesian closed ones, Day tensor products, tensor products of many algebraic structures, …

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- *Non-symmetric bi-closed*: endomorphism categories of the bi-categories of profunctors, relations, V-matrices, spans, …

Non-symmetric *by convention*

- *Non-symmetric one-sided-closed*: endofunctor and composition, lexicographic products, semi-direct products [Ful16]

Non-symmetric *by nature*

Type Formers à la Carte

We don't want a linear language with a fixed set of type formers, but a menu for *type formers à la carte*:

- Ordered linear contexts
- Exchangeable linear contexts
- Linear products/functions
- Cartesian products/functions
- Coproducts
- Fixed-point types
- Linear/non-linear adjunctions
- …

Typing rules for these are standard [Ben95, PP99, Lam58, YW26, JM10].

Type Formers à la Carte

I am working on a Haskell library `linearity-lab` for linear type formers à la carte.

- all aforementioned type formers and the semantics in *Functor* are implemented;
- a programming library, *not a formalisation* – general recursion, recursive types, no equational laws.
- only ordered contexts and zero-or-one-variable contexts (as in CBPV) for now, but exchangeable contexts and graded contexts are planned for future.

Implementation techniques are heavily insipired by [PZ17].

Implementation Design Choices

Design Choice

Type formers are formulated as Haskell type-classes.

For example, the two rules for ordered linear contexts (variable and substitution)

$$\frac{}{x : t \vdash x : t} \qquad \frac{\Delta \vdash M : t_1 \qquad \Gamma_l, x : t_1, \Gamma_r \vdash N : t_2}{\Gamma_l, \Delta, \Gamma_r \vdash \mathit{let}\ x = M\ \mathit{in}\ N : t_2}$$

becomes

$$\begin{aligned}
&\textbf{class}\ \mathit{HasOrdCtx}\ (\mathit{tm} :: \mathit{LCtx} \to \mathit{LType} \to \mathit{Type})\ \textbf{where} \\
&\quad \mathit{var} :: \mathit{tm}\ [(x, t)]\ t \\
&\quad \mathit{llet_} :: \mathit{tm}\ \mathit{ctx}\ t_1 \to \mathit{tm}\ (\mathit{ctxL} +\!\!+ [(x, t_1)] +\!\!+ \mathit{ctxR})\ t_2 \\
&\qquad \to \mathit{tm}\ (\mathit{ctxL} +\!\!+ \mathit{ctx} +\!\!+ \mathit{ctxR})\ t_2
\end{aligned}$$

Implementation Design Choices

Design Choice

Use *numerically named variables and equality constraints* for better type inference.

With named variables, we can compute the left and right contexts $ctxL, ctxR$ from the middle context ctx and the whole context ctx'. We define

$$
\begin{aligned}
&\mathit{llet} :: (\mathit{HasCtx}\ tm, ctx' \sim ctxL +\!\!+ ctx +\!\!+ ctxR, (ctxL, ctxR) \sim \mathit{Split}\ ctx\ ctx') \\
&\quad \Rightarrow tm\ ctx\ t_1 \\
&\quad \rightarrow (tm\ [(\mathit{Fresh}\ ctx', t_1)]\ t_1 \rightarrow tm\ (ctxL +\!\!+ [(\mathit{Fresh}\ ctx', t_1)] +\!\!+ ctxR)\ t_2) \\
&\quad \rightarrow tm\ ctx'\ t_2 \\
&\mathit{llet}\ x\ body = \mathit{llet\_}\ @tm\ @ctxL\ @ctx\ @ctxR\ n\ x\ (body\ var)
\end{aligned}
$$

Implementation Design Choices

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Use *numerically named variables and equality constraints* for better type inference.

Consequently supposing $u :: tm\ [(1, B)]\ D$, we don't have to write

$$t :: tm\ [(0, A), (1, B), (2, C)]\ E$$
$$t = llet\_\ @[(0, A)]\ @[(2, C)]\ u\ (\lambda b \to \ldots)$$

because *llet* can compute $Split\ [(1, B)]\ [(0, A), (1, B), (2, C)] = ([(0, A)], [(2, C)])$

$$t :: tm\ [(0, A), (1, B), (2, C)]\ E$$
$$t = llet\ u\ (\lambda b \to \ldots)$$

Without uniquely named variables this can't be done.

Implementation Design Choices

Design Choice

Variable terms are passed around at the meta level.

We can write terms like $llet\ u\ (\lambda b \longrightarrow b)$ because

$$
\begin{aligned}
&llet :: (\ldots) \Rightarrow tm\ ctx\ t_1 \\
&\quad \longrightarrow (tm\ [(Fresh\ ctx', t_1)]\ t_1 \longrightarrow tm\ (ctxL +\!\!+ [(Fresh\ ctx', t_1)] +\!\!+ ctxR)\ t_2) \\
&\quad \longrightarrow tm\ ctx'\ t_2
\end{aligned}
$$

It feels like programming with HOAS, but under the hood there is no magic:

$$llet\_ :: tm\ ctx\ t_1 \longrightarrow tm\ (ctxL +\!\!+ [(x, t_1)] +\!\!+ ctxR)\ t_2 \longrightarrow tm\ (ctxL +\!\!+ ctx +\!\!+ ctxR)\ t_2$$

Some Example Programs

Define $L\ a$ as the fixed-point $L\ a \cong One + (a \otimes L\ a)$:

$$\textbf{type}\ L :: LType \longrightarrow LType \qquad \textbf{data}\ L\_\ (a :: LType)$$
$$\textbf{type}\ L\ (a :: LType) = LFix\ (L\_\ a) \qquad \textbf{type instance}\ Decode\ (L\_\ a)\ x = One + (a \otimes x)$$

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The type formers we need for lists:

$$\textbf{type}\ HasList\ tm = (HasOrdCtx\ tm, HasLFun\ tm, HasLFix\ tm, HasLProd\ tm, HasCoProd\ tm)$$

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Programming in the order linear language isn't too different from ordinary Haskell:

$$cayley :: (HasList\ tm, \ldots) \Rightarrow tm\ ctx\ (L\ a) \to tm\ ctx\ (L\ a \multimap L\ a)$$
$$cayley\ xs = matchL\ xs\ (abs\ \$\ \lambda x \to x)\ (\lambda x\ xs \to abs\ (\lambda ys \to consL\ x\ (cayley\ xs\ \hat{}\ ys)))$$
$$app :: (HasList\ tm, \ldots) \Rightarrow tm\ ctx_1\ (L\ a) \to tm\ ctx_2\ (L\ a) \to tm\ ctx\ (L\ a)$$
$$app\ xs\ ys = cayley\ xs\ \hat{}\ ys$$

Some Example Programs

We obtain free monads by interpreting lists in the endofunctor category:

$$\begin{aligned}
&\textbf{type}\ \mathit{Free} :: (\mathit{Type} \to \mathit{Type}) \to (\mathit{Type} \to \mathit{Type}) \\
&\textbf{type}\ \mathit{Free}\ f = \mathsf{Fct}[\![(L\ (\mathit{Fct}\ f))]\!] \\
&\mathit{injFree} :: \forall f\ x.\ \mathit{Functor}\ f \Rightarrow f\ x \to \mathit{Free}\ f\ x \\
&\mathit{injFree} = \mathit{asFctMap}\ @[\mathit{Fct}\ f]\ \$\ \lambda(\mathit{Vars}\ c) \to \mathit{singL}\ c \\
&\mathit{joinFree} :: \forall f\ x.\ \mathit{Functor}\ f \Rightarrow \mathit{Free}\ f\ (\mathit{Free}\ f\ x) \to \mathit{Free}\ f\ x \\
&\mathit{joinFree} = \mathit{asFctMap}\ @[L\ (\mathit{Fct}\ f), L\ (\mathit{Fct}\ f)]\ \$\ \lambda(\mathit{Vars}\ (c, k)) \to \mathit{appL}\ c\ k
\end{aligned}$$

Along this line, we can interpret *CList* to obtain an efficient implementation of free monads (not shown here for the lack of time).

Compare with a Manual Translation

Without going through the abstraction of linear logics and monoidal categories, I don't think I can come up with *CListF* directly:

```
data Free f a where
  Ret :: a → Free f a
  Cons :: f a → (a → Free f b) → Free f b
```

```
data SnocF f a where
  Ret' :: a → SnocF f a
  Snoc :: SnocF f a → (a → f b) → SnocF f b
```

```
data QueueF f a where
  Q :: Int → Free f a → Int → (a → SnocF f b) → QueueF f b
```

```
data CListF f a where
  E :: a → CListF f a
  C :: f b → (b → QueueF (CListF f) a) → CListF f a
```

Application

CListF is useful for abstract syntax with efficient substitution ($O(1)$-per-variable).

- Substitution of free monads is *not capture-avoiding*. If we want capture-avoiding, we can work in the category of nominal sets.
- An efficient implementation of name abstraction of nominal sets $[\mathbb{A}]X$ is via stateful fresh-name generation.

Note however, in some situations efficient substitution can be implemented more easily, such as having an 'environemnt' [Yan26].

Conclusion

Things I didn't talk about today:

- a careful complexity analysis of *CListF*
- the need of memoization for $(f \circ g)\ y = \exists x.\, f\ x \times (x \longrightarrow g\ y)$
- embedding functors in profunctors for performance

Future directions:

- Stateful algorithms in moniodal categories
- Laziness in moniodal categories
- Algorithmic complexity in moniodal categories

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